System theory of the integrating sphere as a model for the wireless optical indoor communication channel

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Abstract: The integrating sphere is the base for a physical model of the wireless optical indoor communication channel. The author has found that the approximate expression for the channel transfer function used in that model can be replaced by an exact expression. A comparison shows that the approximation is valid for a wide range of interest, however there are some differences for high frequency values. For the impulse response (obtained by the inverse Fourier transform), the author observes significant deviations in the time region where first order reflections contribute significantly to the curve shape.

1 Introduction

Wireless optical transmission is a very attractive option for indoor communication systems [1]. One of its benefits is that frequency planning is not necessary because optical signals do not go through walls. To evaluate the performance of an optical transmission scheme, a model for the optical channel is needed. A physical model of the indoor wireless optical communication channel was proposed by Jungnickel et al. [2]. This model based on the properties of the integrating sphere (also called Ulbricht sphere) is quite useful when a simple analytical formula for a rough characterisation of the channel is required (see e.g. [3–5]).

The idea of the heuristic model is to use an analytic expression for the channel impulse response (or the transfer function) of the integrating sphere that depends on the surface area and the volume of the sphere and then to replace these quantities by the surface area and the volume of the room. Even though this approach is obviously quite a rough approximation of the reality, it may serve as a valuable rule of thumb for the optical channel of a rectangular room. Even though the model is not able to cover all details, it may come surprisingly close to reality for some scenarios [2, 6].

The above-mentioned analytic expression for the impulse response of the integrating sphere channel was discovered by Pohl et al. [7]. Their paper also presents some Monte–Carlo simulations of the channel in order to examine the model and to justify the approximations that are made in its derivation.

In this paper, we present a novel frequency domain approach that allows to find an exact analytic solution for the channel transfer function of the integrating sphere with an isotropic light source located in the centre of the sphere. It is thus easy to examine the analytical formula presented [7] without performing simulations. The numerical evaluation of both analytic expressions show a very good agreement in the region of interest. However, for high frequencies (corresponding to small delay times), there are deviations that are obviously due to the approximations made in [7] in order to find a simple model.

It is worth mentioning that our analytical treatment is done in the frequency domain for the channel transfer function rather than in the time domain for the channel impulse response. This approach allows us to derive a novel generic integral equation for the location-dependent transfer function inside an arbitrary cavity. The solution of this integral equation can be found analytically for the integrating sphere. For other cavities, the equation must be solved numerically to obtain the transfer function. The impulse response can then be calculated numerically from the transfer function by means of the inverse fast Fourier transform (IFFT).

This equations says that the irradiance $E(x)$ is the sum of two contributions: The first term on the right hand side, $E_{\text{LOS}}(x)$, is the irradiance originating from the line-of-sight (LOS) path, i.e. the direct light from the active source. The irradiance described by the second term is an integral over the surface elements $dA(x')$ of the cavity that act as passive light sources due to their reflectivity. In particular, $\rho(x')E(x')dA(x')$ is the power that is back-reflected from the surface element $dA(x')$, where $\rho(x')$ is the reflectivity at that position $x'$. We assume that the surface elements are ideal Lambertian reflectors [1], which means that the path loss factor between source position $x'$ and receiving position $x$ is given by the Lambertian integral kernel [9]

$$K(x, x') = \frac{1}{\pi} \frac{\cos \theta(x, x') \cos \eta(x, x')}{d(x, x')} V(x, x').$$
The path loss factor of the LOS path is defined similarly as one of the angles (including all reflections) from the active transmitter (defined by the ratio \( \frac{\rho(x,x')}{d(x,x')} \)), and the angle of incident at position \( x \) of the impinging light ray from \( x' \) to \( x \) is denoted by \( \psi(x,x') \). We write

\[
d(x,x') = |x - x'| \tag{3}
\]

for the distance between \( x \) and \( x' \). The emission and the reception are hemispherically orientated, which means that \( K(x,x') = 0 \) if one of the angles \( \theta(x,x') \) or \( \psi(x,x') \) exceeds \( \pi/2 \). This property is included in the visibility factor \( V(x,x') \) which is equal to one if the transmitter and the receiver are visible to each other, and it is equal to zero if not.

In (1), we distinguish between the active light source which is so small that it can be assumed neither to absorb nor to reflect any light and the passive reflectors at the surface that do not emit light by themselves. Keeping this in mind and replacing the irradiance \( E(x) \) by the radiosity \( B(x) = \rho(x)E(x) \), we see that (1) is equivalent to the radiosity equation \([8, 9]\) that is well-known in computer graphics.

The path loss factor \( \eta(x) \) between a light source that emits the power \( \Phi_{ts} \) and the irradiance \( E(x) \) on the surface at position \( x \) is defined by the ratio

\[
\eta(x) = \frac{E(x)}{\Phi_{ts}}. \tag{4}
\]

The path loss factor of the LOS path is defined similarly as

\[
\eta_{\text{LOS}}(x) = \frac{E_{\text{LOS}}(x)}{\Phi_{ts}}. \tag{5}
\]

We note that the dimension of \( \eta(x) \) is \( m^{-2} \). This is due to the fact that the path loss factor describes the path loss per unit area at the receiving position of the impinging light. We may thus speak of \( \eta(x) \) as a path loss density. Using the definition of (4) and (5), (1) can be converted to the following integral equation for the path loss factor:

\[
\eta(x) = \eta_{\text{LOS}}(x) + \int_{\mathcal{S}} K(x,x') \psi(x,x') \eta(x') dA(x'). \tag{6}
\]

### 2.2 System theory: impulse response and transfer function

We consider the optical communication channel for intensity modulation and direct detection (IM/DD). Intensity modulation means that the information signal is modulated directly on the time-variant transmitted optical power \( s(t) \) which is a real-valued and non-negative baseband signal.

For the cavity with a light source inside, the optical channel (including all reflections) from the active transmitter (Tx) to the receiving position at an infinitesimally small area element \( dA(x) \) on the inner surface of the cavity is modelled as a LTI system with a real-valued and non-negative impulse response \( dh(t; x) \geq 0 \). We write \( dh(t; x) \) because it belongs to an infinitesimally small receiver of size \( dA(x) \). The corresponding received signal is given by [1]

\[
dr(t; x) = dh(t; x) * s(t). \tag{7}
\]

The Fourier transform

\[
dH(f; x) = \int_{-\infty}^{\infty} e^{-j2\pi ft} dh(t; x) dt \tag{8}
\]

of \( dh(t; x) \) is the (infinitesimal) transfer function of this channel. We define the transfer function density \( H(f; x) \) by

\[
H(f; x) = H(f; x) dA(x). \tag{9}
\]

The transfer function for a receiver extended to a finite area \( dA \) can thus be obtained as the surface integral

\[
H(f; dA) = \int_{dA} H(f; x) dA(x). \tag{10}
\]

Denoting the statistical average by \( E\{\cdot\} \), we conclude from (7) and (8) that the average optical transmitted and the (infinitesimal) received power

\[
\Phi_{ts} = E\{s(t)\} \quad \text{and} \quad \Phi_{rs} = E\{dr(t; x)\}, \tag{11}
\]

respectively, of the channel are related by

\[
d\Phi_{rs}(x) = dH(0; x) \Phi_{ts}. \tag{12}
\]

Because of the above equation, we can interpret

\[
d\Phi(x) = dH(0; x) \Phi(x),
\]

as the differential optical path loss factor for the infinitesimal area element \( dA(x) \). The path loss factor \( \eta(x) \) of (6) is thus related to the transfer function density \( H(f; x) \) by

\[
\eta(x) = H(0; x). \tag{13}
\]

We note that, due to Fourier transform properties, the inequality \( |H(f)| \leq H(0) \) holds for all frequencies \( f \). The latter property reflects the low-pass behaviour of the optical IM/DD channel.

### 2.3 Basic integral equation

We now combine the perceptions of Sections 2.1 and 2.2 to establish an integral equation for the transfer function density \( H(f; x) \). We note that the LOS path for any (active or passive) source inside the cavity has a transfer function density

\[
H(f) = \eta e^{-j2\pi ft}. \tag{14}
\]

of a simple delay term for the path runtime \( \tau \) multiplied by the attenuation factor \( \eta \) of the corresponding path loss density. Since the Lambert kernel \( K(x,x') \) defined by (2) is the path loss density corresponding to the LOS link from \( x' \) to \( x \), it corresponds to the transfer function density defined by

\[
K(f; x, x') = K(x, x') e^{-j2\pi f \tau(x,x')}. \tag{15}
\]

where

\[
\tau(x, x') = \frac{d(x, x')}{c_0} = \frac{|x - x'|}{c_0} \tag{16}
\]

is the runtime from \( x' \) to \( x \), and \( c_0 \) is the velocity of light. For the LOS path from the source to the surface position \( x \), we must set

\[
H_{\text{LOS}}(f; x) = \eta_{\text{LOS}} e^{-j2\pi f \tau(x, \text{Tx})}, \tag{17}
\]

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where \( \tau(x,Tx) \) is the runtime between the light source and the position \( x \).

Using these definitions, we generalise the path loss integral equation (6) by replacing the kernel \( K(x,x') \) by the kernel \( K(f;x,x') \) and the path loss factors \( \eta(x) \) and \( \eta(x') \) by the transfer function densities \( H(f;x) \) and \( H_{LOS}(f;x) \), respectively. As a result, the transfer function density for the light on the surface can be established by the integral equation

\[
H(f;x) = H_{LOS}(f;x) + \int D \cdot K(f;x,x') \rho(x') \rho f H(f;x') dA(x') . \tag{18}
\]

This equation describes the generic transfer function density for an arbitrary cavity with an inner surface of Lambertian reflectors. A discrete approximation of the integral equation has already been formulated by the vector-matrix equation (29) in [6], where a discrete reflector model is introduced by dividing the surface into small elements. In the general case, such a discretisation is necessary to solve (18) numerically. However, as shown in the following section, there is a special case where the integral equation can be solved analytically.

### 3 Analytic solution for the integration sphere

We consider the case where the cavity is given by an integrating sphere of diameter \( D \) as depicted in Fig. 2. The inner surface has constant reflectivity \( \rho \), and it is illuminated by an isotropic light source located at the centre of the sphere. The perfect spherical symmetry of this setup allows for an analytic solution of the integral equation (18).

#### 3.1 Simplifications for the integral equation

Due to the geometry of the sphere as shown in Fig. 2, the following simplifications apply: For two locations \( x \) and \( x' \) on the sphere, the angle of emission and the angle of incident are identical:

\[
\theta(x,x') = \psi(x,x') . \tag{19}
\]

Furthermore, the distance between the two points \( x \) and \( x' \) is given by

\[
d(x,x') = D \cos \psi(x,x') . \tag{20}
\]

Thus, the Lambert kernel \( K(x,x') \) of (2) can be written as

\[
K(x,x') = \frac{1}{A_{sphere}} \tag{21}
\]

where \( A_{sphere} = \pi D^2 \) is the surface area of the sphere. The kernel \( K(f;x,x') \) of (15) is then given by

\[
K(f;x,x') = \frac{e^{-j\alpha f(x,x')}}{A_{sphere}} . \tag{23}
\]

The assumption of an isotropic light source at the centre leads to

\[
H_{LOS}(f;x) = \frac{e^{j\alpha fT}}{A_{sphere}} , \tag{24}
\]

where

\[
T = \frac{D}{\tau_0} \tag{25}
\]

is the runtime of the light across the diameter of the sphere. We may write \( H_{LOS}(f) \equiv H_{LOS}(f;x) \) because \( H_{LOS}(f;x) \) in (24) is independent of \( x \). Thus, (18) simplifies to

\[
H(f;x) = H_{LOS}(f) + \rho \int D \cdot \frac{e^{-j\alpha f(x,x')}}{A_{sphere}} H(f;x') dA(x') . \tag{26}
\]

As a consequence of the rotational symmetry of the setup, the solution \( H(f;x) \) of this integral equation is rotationally invariant, i.e. it does not depend on the location vector \( x \) on the sphere. A prove of this intuitively obvious fact is given in Appendix 7.1. We may thus write \( H(f) \equiv H(f;x) \) in (26), and the integral equation turns into the algebraic equation

\[
H(f) = H_{LOS}(f) + \rho P(f) H(f) \tag{27}
\]

where we have defined the sphere-to-sphere transfer function

\[
P(f) \equiv \int D \cdot \frac{e^{-j\alpha f(x,x')}}{A_{sphere}} dA(x') . \tag{28}
\]

We can now interpret the transfer function density \( H(f) \) on the surface of the sphere given by (27) as the sum of two contributions: The first contribution corresponds to the LOS component from the source, and the second corresponds to the scattering from the surface back to the surface. This component is proportional to the transfer function density \( H(f) \) itself, it is attenuated by the reflectivity factor \( \rho \) and modified by the transfer function from \( P(f) \) from surface to surface, which gives rise to the name sphere-to-sphere transfer function. We observe that (27) is a simple scalar analogue to the vector-matrix (29) in [6].

The recursion for \( H(f) \) stated in (27) can be resolved to give the explicit expression

\[
H(f) = \frac{H_{LOS}(f)}{1 - \rho P(f)} . \tag{29}
\]

Writing this as a geometric series

\[
H(f) = H_{LOS}(f) + \rho P(f) H_{LOS}(f) + (\rho P(f))^2 H_{LOS}(f) + \ldots \tag{30}
\]

we can associate the term with index \( m \) to all the contributions of the reflections of order \( m \): The first term \( (m=0) \) corresponds to the LOS path \( Tx \rightarrow \text{surface} \), the second term \( (m=1) \) corresponds to first order reflections \( Tx \rightarrow \text{surface} \rightarrow \text{surface} \), the third term \( (m=2) \) corresponds to second order reflections \( Tx \rightarrow \text{surface} \rightarrow \text{surface} \rightarrow \text{surface} \), and so on. This again
substantiates the interpretation of \( P(f) \) as the sphere-to-sphere transfer function \( \text{surface} \rightarrow \text{surface} \).

Switching to the time domain, the impulse response density \( h(t) \) corresponding to (30) can be written as

\[
h(t) = h_{\text{LOS}}(f) + \rho p(t) * \rho p(t) * h_{\text{LOS}}(t) + \rho^2 p(t) * \rho p(t) * h_{\text{LOS}}(t) + \ldots
\]  

(31)

Again, the terms can be associated with the reflections of increasing order, and

\[
p(t) * - - - \cdot \cdot \cdot P(f)
\]  

(32)

can be interpreted as the sphere-to-sphere impulse response. However, the infinite sum of multiple convolutions given by (31) is less convenient for the practical evaluation than the equivalent frequency-domain expression of (29).

3.2 Calculation of the spherical transfer function

In the integral of (28) for \( P(f) \), the parameter \( x \) corresponds to an arbitrary but fixed point on the sphere. The value of the integral does not depend on the choice of this point. As shown in Appendix 7.2, the integral can be written as the Fourier transform of

\[
p(t) \overset{\Delta}{=} \begin{cases} \frac{2t}{T} : 0 \leq t \leq T \\ 0 : \text{elsewhere} \ . \end{cases}
\]  

(33)

This sphere-to-sphere impulse response \( p(t) \) is depicted in Fig. 3. We note that \( p(t) \) can be interpreted as the probability density function (pdf) of the photon runtimes between surface elements inside the sphere. The Fourier transform of \( p(t) \) evaluated in Appendix 7.2 results in

\[
P(f) = e^{-j\pi f T} \left( \frac{t}{T} + \frac{j}{2} \text{sinc}(fT) \right).
\]  

(34)

The real and imaginary part and the absolute value of this sphere-to-sphere transfer function are depicted in the upper plot of Fig. 4. We observe that

\[
P(0) = 1
\]  

and

\[
\text{Re}\{P(f)\} = 0 \text{ for } fT = 1, 2, 3, \ldots
\]  

(35)  

and

(36)

The lower plot in Fig. 4 shows the Nyquist curve of \( P(f) \) in the complex plane.

4 Numerical evaluation of the formula

4.1 Definition of the diffuse component

In the following, we drop the numerator \( e^{-j\pi f T} \) in (24) and write

\[
H_{\text{LOS}}(f; x) = \frac{1}{A_{\text{sphere}}}. \]

(37)

This simply means that we measure the time \( t \) relative to the time instant when the LOS component reaches the sphere. Without losing generality, we may consider the whole sphere \( \mathcal{S} \) as a receiver and write

\[
H(f; \mathcal{S}) = A_{\text{sphere}} H(f)
\]  

(38)

for its transfer function given by (10). For any smaller receive area \( A_{\mathcal{R}} \) on the surface, the results have simply to be scaled by \( A_{\mathcal{R}} / A_{\text{sphere}} \). Inserting (37) and (38) into (29), we obtain

\[
H(f; \delta) = \frac{1}{1 - \rho P(f)}.
\]  

(39)

This is the sum of a very simple LOS component

\[
H_{\text{LOS}}(f; \delta) = 1
\]

and a diffuse component is given by

\[
H_{\text{diff}}(f; \delta) = \frac{\rho P(f)}{1 - \rho P(f)}.
\]  

(40)

We note that
We shall compare our exact formula with the simplified analytical IFFT. It can be calculated numerically from

\[
\text{values } \eta(\varphi) \text{ and } \rho \rangle \text{ to the total received power.}
\]

The impulse response \( h_{\text{diff}}(f; \varphi) \) corresponding to \( H_{\text{diff}}(f; \varphi) \) is defined by

\[
h_{\text{diff}}(f; \varphi) = \frac{\eta_{\text{diff}}(\varphi)}{1 + \frac{1}{\rho}}. \tag{43}
\]

It can be calculated numerically from \( H_{\text{diff}}(f; \varphi) \) by means of the IFFT.

4.2 Simplified model of Pohl et al. [7]

We shall compare our exact formula with the simplified analytical model of Pohl et al., [7] which approximates the diffuse optical impulse response for the integrating sphere by the exponential

\[
h_{\text{diff}}(t; \varphi) = \frac{\eta_{\text{diff}}(\varphi)}{\tau} e^{-\frac{t}{\tau}} \epsilon(t). \tag{44}
\]

where \( \epsilon(t) \) is the jump function, and \( \eta_{\text{diff}}(\varphi) \) is given by (41). The time constant

\[
\tau = \frac{2}{3 \ln \rho} \tag{45}
\]

can be interpreted as the average lifetime of a photon inside the sphere until it is absorbed by the inner surface. This time constant is obtained from the model assumption in [7] that replaces the photon runtime

\[
t_{\text{ave}}(\vartheta, \varphi) = T \cos \vartheta = \frac{D}{c_0} \cos \vartheta, \tag{46}
\]

between two reflections by its average value

\[
t_{\text{ave}} = \frac{2}{3} T = \frac{2 D}{3 c_0} \tag{47}
\]

obtained from the pdf given by (33). This can be written as

\[
t_{\text{ave}} = \frac{4 \text{V}_{\text{sphere}}}{A_{\text{sphere}} c_0} \tag{48}
\]

where \( \text{V}_{\text{sphere}} \) is the volume of the sphere. The physical model of the optical indoor channels described in [2] replaces \( A_{\text{sphere}} \) and \( \text{V}_{\text{sphere}} \) by the corresponding parameters of an rectangular room.

The transfer function corresponding to (44) is given by the simple first-order low-pass filter transfer function

\[
H_{\text{diff}}(f; \varphi) = \frac{\eta_{\text{diff}}(\varphi)}{1 + \frac{1}{\rho}} \tag{49}
\]

As we shall see below, this is a not only a simple but also a sufficiently accurate approximation for the exact analytical solution given by (40), – at least in the frequency region of interest. In the following, we shall compare the results obtained from both expressions to estimate the range of validity of the approximation.

4.3 Diffuse transfer function

The transfer function has been calculated for different reflectivity values \( \rho \) according to the exact expression in (40) and according to the simplified model of (49). To compare the transfer functions for different values of \( \rho \) in the same plot, it is convenient to plot \( H_{\text{diff}}(f; \varphi)/H_{\text{diff}}(f) \) rather than \( H_{\text{diff}}(f; \varphi) \). The upper plot in Fig. 5 shows this quantity on a linear scale. In the (low-) frequency region shown in that plot, the approximation (dashed lines) is very close to the exact solution (solid lines), especially for reflectivity values close to one. The lower plot for \( H_{\text{diff}}(f; \varphi)/H_{\text{diff}}(f) \) on a decibel scale shows deviations up to 2 dB between the exact and the approximate expression at higher frequencies and for the whole range of reflectivity values. When looking at the complex Nyquist curve depicted in Fig. 6, we see that these deviations correspond to significant phase rotations for \( f \to \infty \) in the exact solution (solid line) that do not occur in the simple first order low-pass filter curve of the approximate model (dashed line).

4.4 Diffuse impulse response

The impulse responses have been calculated numerically by means of the IFFT of the exact expression in (40). To analyse the influence of different reflection orders \( m \) corresponding to the terms in the sum

\[
h_{\text{diff}}(t; \varphi) = \rho p(t) + \rho^2 p(t) \ast p(t) + \rho^3 p(t) \ast p(t) \ast p(t) + \ldots,
\]

their contributions are shown in Fig. 7 for the example \( \rho = 0.8 \). For small times \( t \leq T \), the curve shape is clearly dominated by first order reflections that are described by the sphere-to-sphere impulse response \( p(t) \) depicted in Fig. 3. As observed in [6] for the impulse

![Fig. 5 Transfer function \( H_{\text{diff}}(f) \equiv H_{\text{diff}}(f; \varphi) \) for the integrating sphere on a linear (upper plot) and on a decibel scale (lower plot) for reflectivity values \( \rho \) between \( \rho = 0.4 \) and \( \rho = 0.9 \). Solid curves: exact solution. Dashed curves: approximation [7].](image-url)
response of a room, the higher-order reflections are very significant for this value of $\rho$. It is not possible to produce the correct (exponential) decay when only considering a low number of reflections as done in [10].

To show the influence of the reflectivity factor $\rho$, the impulse responses for different values of $\rho$ are depicted in Fig. 8 on a linear scale in the upper plot and on a logarithmic scale in the lower one. As one can read from these plots, the first order reflections determine the curve shapes for $t \leq T$, and they are especially dominant for low values of $\rho$. Obviously, the exponential model of (44) cannot appropriately describe the shapes of the curves for $t \leq T$. To obtain a better fit for $t \gg T$ when comparing with the exact solution, we heuristically introduced a time delay of $t_{\text{delay}} = T/2$ into the exponential model of (44). Such a delay is reasonable because for physical reasons the impulse response must start at $h_{\text{diff}}(0) = 0$ rather than with its maximal value $h_{\text{diff}}/\tau$ as predicted by (44). After introducing this delay, we observe that the curves of the approximate expression (dashed) are nearly identical to the curves of the exact expression (solid) for $t \gg T$. From the logarithmically scaled curves in the lower plot one can read that the exponential model predicts the correct decay of the impulse responses for $t \to \infty$.

5 Conclusion

In this paper, a new and exact analytical expression for the transfer function of the light propagation inside an integration sphere is presented. This improves an approximate expression previously reported in the literature [7] which is the basis of the Jungnickel model [2] for the indoor wireless infrared communication channel. The new idea of our approach is to tackle the problem in the frequency domain. This is much simpler than the usual time-domain approach because convolutions can be replaced by multiplications. We are able to derive an integral equation that describes the transfer function density for each possible receiving position inside a cavity. Our equation is closely related to the radiosity equation known from computer graphics, but it is a generalisation of that equation in the sense that it introduces the frequency variable. To solve the integral equation for arbitrary cavities, numerical methods may be applied. However, for the special but interesting case of the integrating sphere, we are able to derive an analytical solution of that equation. This allows to examine the simple approximative formula presented by Pohl et al. [7]. We found a good overall agreement except for very high frequencies. Switching to the time domain, we observe deviations in the approximate impulse response for small delay times in the order of the runtime that a photon needs to cross the sphere. This can be understood by the fact that the approximations in [7] replace the exact runtimes by an average. For higher delay times, however, the impulse responses of the exact and the approximate formula, respectively, apparently show the same exponential decay.

6 References

7 Appendix

7.1 Proof of the rotational invariance

To prove the rotational invariance of the transfer function density \( H(f; x) \) given by the solution of (26), we consider a 3D rotation matrix \( R \) and the corresponding rotated transfer function density defined by

\[
H_R(f; x) = H(f; R^{-1} x). \tag{50}
\]

To prove the identity \( H_R(f; x) = H(f; x) \), we have to show that \( H_R(f; x) \) obeys the same integral equation (26) as \( H(f; x) \). Assuming that \( H(f; u) \) fulfils the integral equation for \( u = R^{-1} x \), we insert (50) and obtain

\[
H_R(f; x) = H_{\text{LOS}}(f) + \frac{\rho}{A_{\text{sphere}}} \int \frac{e^{-j\pi f(x' - x)}}{\delta} H(f; u) dA(u'). \tag{51}
\]

Just for convenience, the integration variable has been denoted by \( u' \). We apply a coordinate transform on this integration variable by writing

\[ x' = Ru'. \]

Because the Jacobian determinant of the (orthogonal) rotation matrix \( R \) equals one, the transform does not change the infinitesimal area element:

\[ dA(x') = dA(u'). \]

Furthermore, according to (16), the runtime is rotationally invariant:

\[ \tau(R^{-1} x, R^{-1} x') = \tau(x, x') \]

Finally, writing

\[ H(f; u') = H(f; R^{-1} x') = H_{\text{g}}(f; x') \]

we obtain the integral equation

\[
H_R(f; x) = H_{\text{LOS}}(f) + \frac{\rho}{A_{\text{sphere}}} \int \frac{e^{-j\pi f(x' - x)}}{\delta} H_{\text{g}}(f; x') dA(x') \tag{52}
\]

for \( H_R(f; x) \) which is the same as (26) for \( H(f; x) \). This is what had to be proven.

7.2 Evaluation of \( P(f) \)

To evaluate the integral of (28), spherical coordinates \((\theta, \phi)\) with respect to the centre of the sphere are introduced, where (without losing generality and as a consequence of the rotational invariance) the receiving point \( x \) can be chosen to be located at the bottom of the sphere as depicted in Fig. 2. The infinitesimal surface element \( dA(x') \) can be expressed by spherical coordinates as

\[
dA(x') = \frac{1}{2} \sin \theta d\theta d\phi \]

\[
= \frac{A_{\text{sphere}}}{4\pi} \sin \theta d\theta d\phi. \tag{53}
\]

As one can see from Fig. 2, the polar angle \( \theta \) is just twice the angle \( \psi \) of the impinging ray:

\[ \theta = 2 \psi. \tag{54} \]

Using the relation

\[ \sin(2\psi) = 2\sin \psi \cos \psi \tag{55} \]

we obtain

\[
dA(x') = \frac{A_{\text{sphere}}}{\pi} \sin \psi \cos \psi d\psi d\phi. \tag{56}
\]

Equation (28) can thus be written as

\[
P(f) = \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-j2\pi f(x'' - x)} \sin \psi \cos \psi d\psi d\phi. \tag{57}
\]

Because the integrand does not depend on \( \phi \), integrating over this variable leads to

\[
P(f) = 2 \int_{-\pi}^{\pi} e^{-j\pi f(x'' - x)} \sin \psi \cos \psi d\psi. \tag{58}
\]

We substitute

\[ \tau = T \cos \psi \tag{59} \]

for the runtime \( \tau = \tau(x, x') \) and obtain

\[
P(f) = \int_{0}^{T} e^{-j\pi f(\tau + 2\pi \tau/T)} d\tau. \tag{59}
\]

This is the Fourier transform of the function

\[
p(t) \triangleq \begin{cases} \frac{2t}{T^2} & : 0 \leq t \leq T \\ 0 & : \text{elsewhere} \end{cases} \tag{60}
\]

depicted in Fig. 3.

We write

\[
p(t) \leftrightarrow \mathcal{F} \{ P(f) \} \tag{61}
\]

for the Fourier transform pair. The rectangle function \( \text{rect}(t/T) \) between \(-T/2\) and \(T/2\) has the Fourier transform

\[
\text{rect}(\frac{T}{2}) \leftrightarrow T \text{sinc}(fT). \tag{62}
\]

We define an auxiliary function

\[
q(t) = \frac{2t}{T^2} \text{rect}(\frac{t}{T}) \tag{63}
\]
that is related to \( p(t) \) by

\[
p(t) = q\left(t - \frac{T}{2}\right) + \frac{1}{T} \text{recl}\left(\frac{t}{T} - \frac{1}{2}\right).
\]

(64)

The Fourier transform of the auxiliary function is given by

\[
q(t) \leftrightarrow \frac{j}{\pi} \text{sinc}'(fT),
\]

(65)

where \( \text{sinc}'(x) \) denotes the first derivative of \( \text{sinc}(x) \). One can easily show:

\[
\text{sinc}'(x) = \begin{cases} 
\frac{1}{(\pi x)^2} (\pi x \cos(\pi x) - \sin(\pi x)) & : x \neq 0 \\
0 & : x = 0 
\end{cases}
\]

(66)

The Fourier transform of \( p(t) \) is then readily obtained from the Fourier transform of \( q(t) \) as

\[
P(f) = e^{-j\pi f T} \left( \text{sinc}(fT) + \frac{j}{\pi} \text{sinc}'(fT) \right).
\]

(67)